

The dispersion relation and quality factor of TM_{01m} mode in standing wave dielectric structure

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1. Introduction

We want to construct and test an 11.4 GHz dielectric structure based standing wave accelerator. The motivation is to use modest RF power to achieve high gradient in a dielectric loaded structure. Due to the available RF source at NRL (2 – 20 MW at 11.424 GHz), we would like to achieve $> 30 \text{ MV/m}$ gradient. In this paper, we calculate the physics properties of a dielectric loaded standing wave accelerator, including field distribution, quality factor Q and the dispersion relation of the TM_{01m} accelerating modes. We also get the longitudinal electric field amplitude E_{zm} along the axis under 3MW input power for different length of such structure.

2. Electrical and Magnetic field distribution in a standing wave dielectric structures.

This structure is a metallic tube with inner radius b , partially filled with isotropic material with dielectric constant ϵ , containing a hole of radius a at the center, shown as Fig. 1.

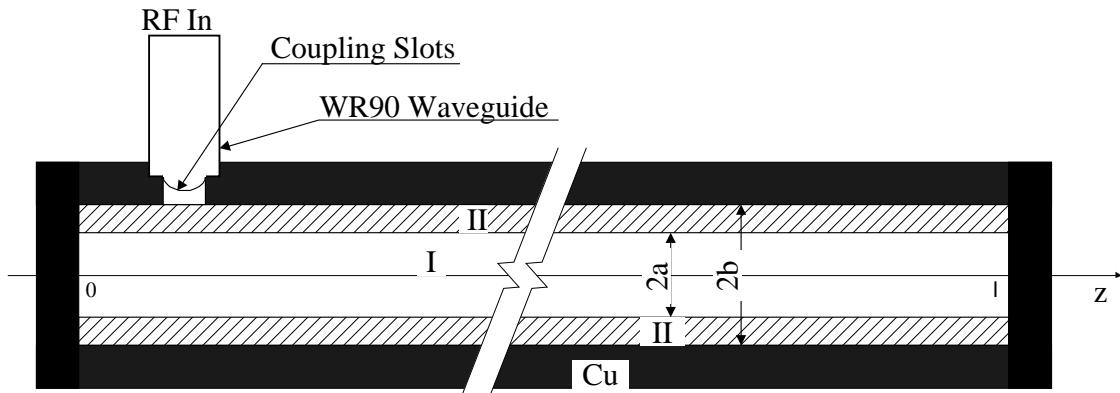


Fig. 1. Dielectric loaded standing wave structure.
 Region I: vacuum, II: dielectric.

The Maxwell's equations for longitudinal fields are

$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right) E_z = 0 \quad (1)$$

$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right) H_z = 0 \quad (2)$$

From Maxwell's equations, the longitudinal fields of TM₀₁ in region I and II are presented by, respectively,

$$\text{I: } \begin{aligned} E_{z1}(r, \varphi, z) &= AJ_0(k_{c1}r) \sin(k_{z1}z + \varphi_1) \\ H_{z1}(r, \varphi, z) &= 0 \end{aligned} \quad (3)$$

$$\text{II: } \begin{aligned} E_{z2}(r, \varphi, z) &= B[Y_0(k_{c2}b) \cdot J_0(k_{c2}r) - J_0(k_{c2}b) \cdot Y_0(k_{c2}r)] \sin(k_{z2}z + \varphi_2) \\ H_{z2}(r, \varphi, z) &= 0 \end{aligned} \quad (4)$$

where

$$\begin{aligned} k_{z1} &= k_{z2} = k_z = \frac{\omega}{v_p} \\ k_{c1}^2 &= \omega^2 \epsilon_1 \mu_1 - k_z^2 = \omega^2 \left(\frac{1}{c^2} - \frac{1}{v_p^2} \right) \\ k_{c2}^2 &= \omega^2 \epsilon_2 \mu_2 - k_z^2 = \omega^2 \left(\frac{\epsilon_{r2}}{c^2} - \frac{1}{v_p^2} \right) \end{aligned} \quad (5)$$

The transverse electric fields \mathbf{E}_t and magnetic fields \mathbf{H}_t in a waveguide can always be expressed in terms of the longitudinal components E_z and H_z . The longitudinal components E_z and H_z are given by Maxwell equations

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \quad (6)$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (7)$$

For the general solution we have

$$\begin{aligned} E_r &= \frac{j}{k_c^2} \left(-j \frac{\partial^2 E_z}{\partial z \partial r} - \frac{\omega\mu}{r} \frac{\partial H_z}{\partial \varphi} \right) \\ E_\varphi &= \frac{j}{k_c^2} \left(-j \frac{\partial^2 E_z}{r \partial z \partial \varphi} + \omega\mu \frac{\partial H_z}{\partial r} \right) \\ H_r &= \frac{j}{k_c^2} \left(\frac{\omega\epsilon}{r} \frac{\partial E_z}{\partial \varphi} + j \frac{\partial^2 H_z}{\partial z \partial r} \right) \\ H_\varphi &= \frac{j}{k_c^2} \left(-\omega\epsilon \frac{\partial E_z}{\partial r} + \frac{1}{r} \frac{\partial^2 H_z}{\partial z \partial \varphi} \right) \end{aligned} \quad (8)$$

For TM₀₁ mode we have

$$\begin{aligned}
E_r &= \frac{1}{k_c^2} \frac{\partial^2 E_z}{\partial z \partial r} \\
E_\varphi &= 0 \\
H_r &= 0 \\
H_\varphi &= -\frac{j}{k_c^2} \omega \epsilon \frac{\partial E_z}{\partial r}
\end{aligned} \tag{9}$$

Knowing E_z and H_z , and the boundary condition,

$$E_{z2}(r = b, \varphi, z) = 0 \tag{10}$$

the transverse electric and magnetic fields of TM₀₁ can be computed by using Eq.(9), i.e.,

I($0 \leq r \leq a$):

$$\begin{aligned}
E_{z1}(r, \varphi, z) &= AJ_0(k_{c1}r) \sin(k_z z + \varphi_1) \\
E_{r1}(r, \varphi, z) &= \frac{k_z}{k_{c1}} AJ'_0(k_{c1}r) \cos(k_z z + \varphi_1) \\
E_{\varphi 1}(r, \varphi, z) &= 0 \\
H_{z1}(r, \varphi, z) &= 0 \\
H_{r1}(r, \varphi, z) &= 0 \\
H_{\varphi 1}(r, \varphi, z) &= -j \frac{\omega \epsilon_0}{k_{c1}} AJ'_0(k_{c1}r) \sin(k_z z + \varphi_1)
\end{aligned} \tag{11}$$

II($a \leq r \leq b$):

$$\begin{aligned}
E_{z2}(r, \varphi, z) &= B[Y_0(k_{c2}b)J_0(k_{c2}r) - J_0(k_{c2}b)Y_0(k_{c2}r)] \sin(k_z z + \varphi_2) \\
E_{r2}(r, \varphi, z) &= \frac{k_z B}{k_{c2}} [Y_0(k_{c2}b)J'_0(k_{c2}r) - J_0(k_{c2}b)Y'_0(k_{c2}r)] \cos(k_z z + \varphi_2) \\
E_{\varphi 2}(r, \varphi, z) &= 0 \\
H_{z2}(r, \varphi, z) &= 0 \\
H_{r2}(r, \varphi, z) &= 0 \\
H_{\varphi 2}(r, \varphi, z) &= -\frac{j \omega \epsilon_0 \epsilon_{r2} B}{k_{c2}} [Y_0(k_{c2}b)J'_0(k_{c2}r) - J_0(k_{c2}b)Y'_0(k_{c2}r)] \sin(k_z z + \varphi_2)
\end{aligned} \tag{12}$$

With the boundary condition

$$\begin{aligned}
E_{r1}(r, \varphi, z = 0) &= E_{r1}(r, \varphi, z = l) \\
E_{r2}(r, \varphi, z = 0) &= E_{r2}(r, \varphi, z = l)
\end{aligned} \tag{13}$$

we obtain

$$\begin{aligned}
\varphi_1 &= \varphi_2 = \frac{\pi}{2} \\
k_z &= \frac{m\pi}{l}, \quad m = 0, \pm 1, \pm 2, \dots
\end{aligned} \tag{14}$$

here we force $v_p = c$, then we can obtain

$$k_z = \frac{\omega}{c} = \frac{m\pi}{l}$$

$$l = \frac{m}{2} \lambda, \quad m = 1, 2, \Lambda . \quad (15)$$

For $v_p = c$ under $f = 11.424\text{GHz}$,

$$l = 1.313\text{cm}, \quad 2.626\text{cm}, \quad \Lambda . \quad (16)$$

With the boundary condition

$$E_{z1}(r = a, \varphi, z) = E_{z2}(r = a, \varphi, z) \quad (17)$$

we obtain

$$\frac{A}{B} = \frac{Y_0(k_{c2}b)J_0(k_{c2}a) - J_0(k_{c2}b)Y_0(k_{c2}a)}{J_0(k_{c1}a)} \quad (18)$$

Then fields in the standing wave structure can be written as

I($0 \leq r \leq a$):

$$\begin{aligned} E_{z1}(r, \varphi, z) &= AJ_0(k_{c1}r)\text{Cos}(k_z z) \\ E_{r1}(r, \varphi, z) &= -\frac{k_z}{k_{c1}} AJ'_0(k_{c1}r)\text{Sin}(k_z z) \\ E_{\varphi 1}(r, \varphi, z) &= 0 \\ H_{z1}(r, \varphi, z) &= 0 \\ H_{r1}(r, \varphi, z) &= 0 \\ H_{\varphi 1}(r, \varphi, z) &= -j \frac{\omega \epsilon_0}{k_{c1}} AJ'_0(k_{c1}r)\text{Cos}(k_z z) \end{aligned} \quad (19)$$

II($a \leq r \leq b$):

$$\begin{aligned} E_{z2}(r, \varphi, z) &= AJ_0(k_{c1}a) \frac{Y_0(k_{c2}b)J_0(k_{c2}r) - J_0(k_{c2}b)Y_0(k_{c2}r)}{Y_0(k_{c2}b)J_0(k_{c2}a) - J_0(k_{c2}b)Y_0(k_{c2}a)} \text{Cos}(k_z z) \\ E_{r2}(r, \varphi, z) &= -\frac{k_z}{k_{c2}} AJ_0(k_{c1}a) \frac{Y_0(k_{c2}b)J'_0(k_{c2}r) - J_0(k_{c2}b)Y'_0(k_{c2}r)}{Y_0(k_{c2}b)J_0(k_{c2}a) - J_0(k_{c2}b)Y_0(k_{c2}a)} \text{Sin}(k_z z) \\ E_{\varphi 2}(r, \varphi, z) &= 0 \\ H_{z2}(r, \varphi, z) &= 0 \\ H_{r2}(r, \varphi, z) &= 0 \\ H_{\varphi 2}(r, \varphi, z) &= -j \frac{\omega \epsilon_0 \epsilon_{r2}}{k_{c2}} AJ_0(k_{c1}a) \frac{Y_0(k_{c2}b)J'_0(k_{c2}r) - J_0(k_{c2}b)Y'_0(k_{c2}r)}{Y_0(k_{c2}b)J_0(k_{c2}a) - J_0(k_{c2}b)Y_0(k_{c2}a)} \text{Cos}(k_z z) \end{aligned} \quad (20)$$

3. Dispersion Relation

With the boundary condition

$$H_{\varphi 1}(r = a, \varphi, z) = H_{\varphi 2}(r = a, \varphi, z) \quad (21)$$

and Eq. (20), we obtain the dispersion relation of this structure

$$\begin{aligned} k_{c2} J'_0(k_{c1}a) [Y_0(k_{c2}b)J_0(k_{c2}a) - J_0(k_{c2}b)Y_0(k_{c2}a)] \\ = \epsilon_{r2} k_{c1} J_0(k_{c1}a) [Y_0(k_{c2}b)J'_0(k_{c2}a) - J_0(k_{c2}b)Y'_0(k_{c2}a)] \end{aligned} \quad (22)$$

where

$$k_z = \frac{\omega}{v_p} = \frac{m\pi}{l}, \quad v_p = \frac{\omega l}{m\pi}, \quad m = 0, \pm 1, \pm 2, \Lambda$$

$$k_{c1}^2 = \omega^2 \epsilon_1 \mu_1 - k_z^2 = \omega^2 \left(\frac{1}{c^2} - \frac{1}{v_p^2} \right)$$

$$k_{c2}^2 = \omega^2 \epsilon_2 \mu_2 - k_z^2 = \omega^2 \left(\frac{\epsilon_{r2}}{c^2} - \frac{1}{v_p^2} \right)$$

The dispersion relation of the standing wave structure is discrete.

For TM_{01m} mode, $f = 11.424\text{GHz}$, $a = 0.3\text{cm}$, $b = 0.4567\text{cm}$, and fixed

$l = \frac{\lambda}{2} \times 8 = 10.50\text{cm}$ and $\epsilon_r = 20$, we obtain the dispersion relation of the SW structure shown as in Fig.2.

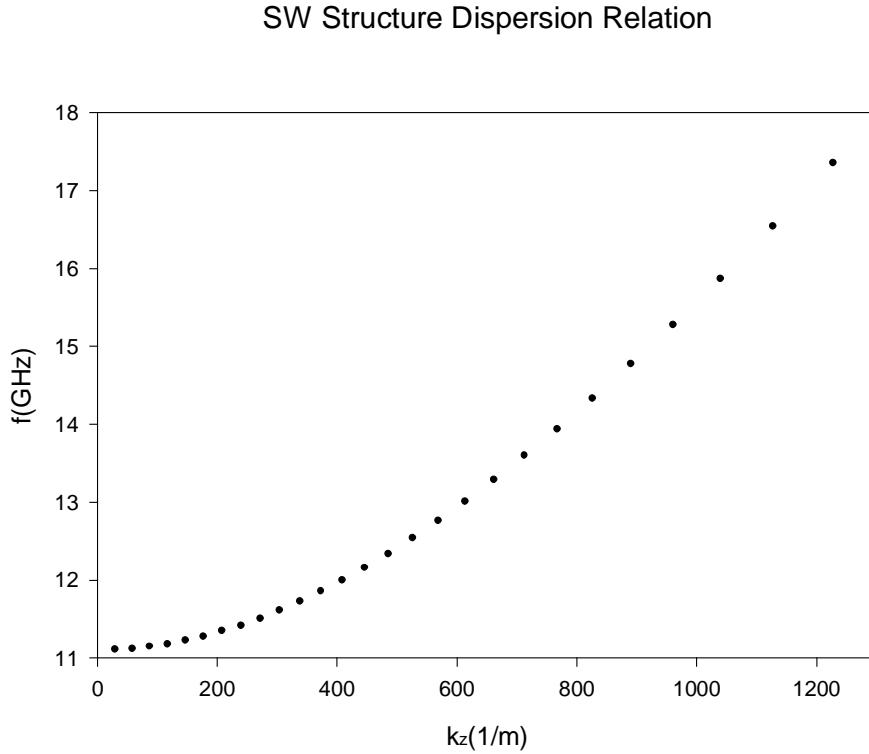


Fig. 2. Calculated TM_{01m} mode dispersion relation of the SW dielectric tube.

$f = 11.424\text{GHz}$, $a = 0.3\text{cm}$, $b = 0.4567\text{cm}$, $l = \frac{\lambda}{2} \times 8 = 10.50\text{cm}$ and $\epsilon_r = 20$.

4. Calculation of the Quality Factor

Define Q is the quality factor of the dielectric standing wave structure,

$$Q = \frac{\omega U}{P} \quad (23)$$

The stored energy in the standing wave structure consist of the one in I and II, or,
 $U = U_I + U_{II}$ (24)

$$\begin{aligned} U_I &= \frac{\epsilon_0}{4} \int_{V_I} \vec{E} \cdot \vec{E}^* dv + \frac{\mu_0}{4} \int_{V_I} \vec{H} \cdot \vec{H}^* dv \\ &= \frac{\epsilon_0}{4} \int_0^l dz \int_0^{2\pi} rd\varphi \int_0^a dr |E_{z1}(r, \varphi, z)| + \frac{\epsilon_0}{4} \int_0^l dz \int_0^{2\pi} rd\varphi \int_0^a dr |E_{r1}(r, \varphi, z)| + \\ &\quad + \frac{\mu_0}{4} \int_0^l dz \int_0^{2\pi} rd\varphi \int_0^a dr |H_{\varphi 1}(r, \varphi, z)| \end{aligned} \quad (25)$$

$$\begin{aligned} U_{II} &= \frac{\epsilon_{r2}\epsilon_0}{4} \int_{V_{II}} \vec{E} \cdot \vec{E}^* dv + \frac{\mu_0}{4} \int_{V_{II}} \vec{H} \cdot \vec{H}^* dv \\ &= \frac{\epsilon_{r2}\epsilon_0}{4} \int_0^l dz \int_0^{2\pi} rd\varphi \int_a^b dr |E_{z2}(r, \varphi, z)| + \frac{\epsilon_{r2}\epsilon_0}{4} \int_0^l dz \int_0^{2\pi} rd\varphi \int_a^b dr |E_{r2}(r, \varphi, z)| + \\ &\quad + \frac{\mu_0}{4} \int_0^l dz \int_0^{2\pi} rd\varphi \int_a^b dr |H_{\varphi 2}(r, \varphi, z)| \end{aligned} \quad (26)$$

$$P = P_{II} + P_{wall} \quad (27)$$

$$\begin{aligned} P_{wall} &= \frac{R_s}{2} \int_s \vec{H}_t \cdot \vec{H}_t^* ds \\ &= \frac{R_s}{2} \int_0^{2\pi} bd\varphi \int_0^l dz |H_{\varphi 2}(r=b, \varphi, z)|^2 + \\ &\quad + 2 \frac{R_s}{2} \left[\int_0^{2\pi} rd\varphi \int_0^a dr |H_{\varphi 1}(r, \varphi, z=0)|^2 + \int_0^{2\pi} rd\varphi \int_a^b dr |H_{\varphi 2}(r, \varphi, z=0)|^2 \right] \end{aligned} \quad (28)$$

$$\text{where } R_s = \sqrt{\frac{\pi f \mu}{\sigma}}$$

Q_{II} is the quality factor of dielectric. Q_{wall} is the quality factor of copper wall.

$$\frac{1}{Q} = \frac{1}{Q_{II}} + \frac{1}{Q_{wall}} \quad (29)$$

For TM_{018} mode, $f = 11.424 GHz$, $v_p = c$, $l = \frac{\lambda}{2} \times 8 = 10.50 cm$, $\epsilon_r = 20$,

$\sigma = 5.8 \times 10^7 S/m$ and $\tan \delta = 10^{-4}$,

$$Q_{II} = \frac{U}{U_{II} \tan \delta_{II}} = 10465, \quad (30)$$

$$Q_{wall} = \frac{\omega U}{P_{wall}} = 2771, \quad (31)$$

$$Q = \frac{1}{\frac{1}{Q_{II}} + \frac{1}{Q_{wall}}} = 2191 \quad (32)$$

From (30) and (31),

$$\frac{P_{wall}}{P_{II}} = \frac{Q_{II}}{Q_{wall}} = 3.8 \quad (33)$$

so the attenuation is dominated by the copper wall losses rather than dielectric losses.

5. Longitudinal Electric Fields along the Axis

In the standing wave structure, input power P_{in} is the same as the power loss on the copper wall and in the dielectric. The coefficient A in Eq. (19), or the longitudinal electric field amplitude E_{zm} along the axis, can be calculated according to Eq. (23)-(26) and (19)-(20).

For different SW structure length l , we calculated Q and E_{zm} under $P_{in} = 3MW$, $f = 11.424GHz$, $v_p = c$, $\epsilon_r = 20$, $\sigma = 5.8 \times 10^7 S/m$ and $\tan \delta = 10^{-4}$. The results are shown in Fig. 3.

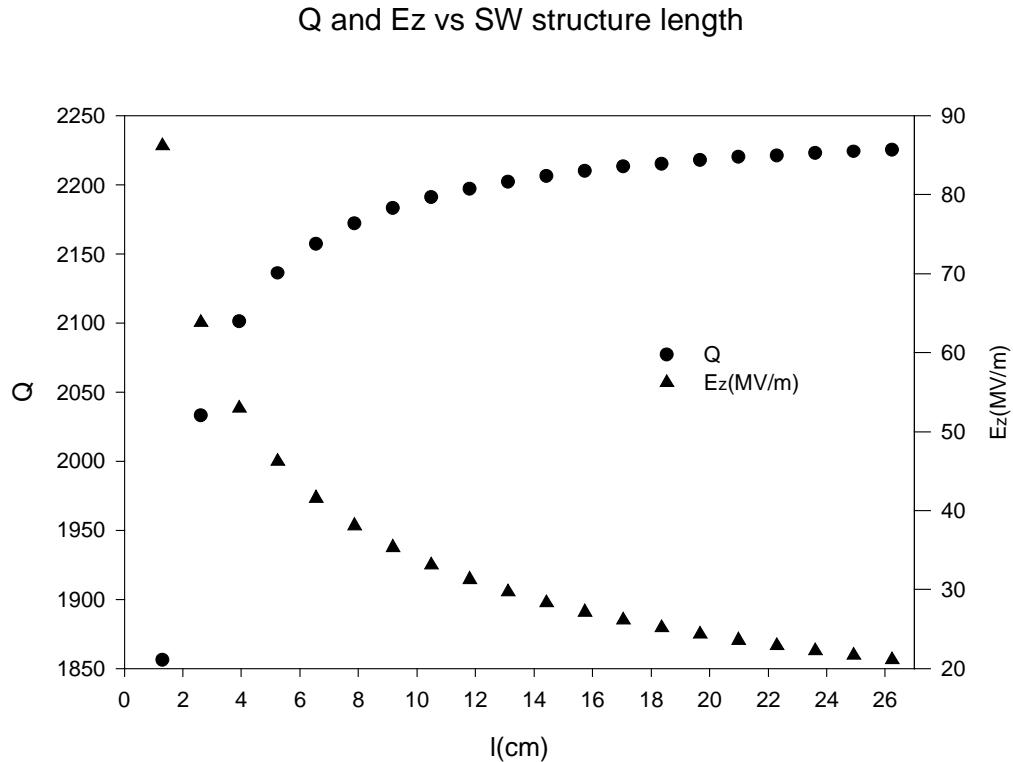


Fig. 3. Calculated Q and E_{zm} for different length of the SW dielectric tube with the parameters as $\epsilon_r = 20$, $\sigma = 5.8 \times 10^7 S/m$ and $\tan \delta = 10^{-4}$ under $P_{in} = 3MW$, $v_p = c$ and $f = 11.424GHz$.

6. Conclusion

We have obtained the electrical and magnetic field distribution, dispersion relation and the quality factor of the dielectric standing wave accelerating structures. The attenuation in dielectric loaded structure at 11.424GHz is dominated by the copper wall losses rather than dielectric losses. We also got the longitudinal electric field amplitude E_{zm} along the axis under 3MW input power for different length of such structure. We found that for this modest amount of RF power, one can obtain 20 – 60 MV/m gradient. This concludes that we should construct a test standing wave accelerator so we can test the high gradient X-band dielectric accelerator at NRL or somewhere else.